.

A BOUND FOR STIELTJES CONSTANTS

S. PAULI AND F. SAIDAK

Abstract. The main goal of this note is to improve the best known bounds for the Stieltjes constants, using the method of steepest descent that was applied in 2011 by Coffey and Knessl in order to approximate these constants.

1. Introduction

Let $\zeta(s)$ denote the Riemann zeta function (for $s \in \mathbb{C}$ defined by Riemann [17] in 1859). The function $\zeta(s)$ has an Euler product (Euler [5] of 1737) and also satisfies a functional equation (Euler [6] of 1749). In this paper we consider the related Hurwitz zeta function $\zeta(s,a)$ (see Hurwitz [11] of 1882), which for $0 < a \leq 1$ has the Laurent series expansion:

$$\zeta(s,a) := \sum_{n=0}^{\infty} \frac{1}{(n+a)^s} = \frac{1}{s-1} + \sum_{n=0}^{\infty} \frac{(-1)^n \gamma_n(a)}{n!}(s-1)^n,$$

where, for non-negative integers n, the coefficients $\gamma_n(a)$ are known as the Stieltjes constants ([20]), which were generalized to *fractional* values $\alpha \in \mathbb{R}^+$ by Kreminski [12] in 2003. These constants have several interesting and unexpected applications in the zeta function theory, as was shown recently in [4], [3], [8], and [9]. Moreover, the classical Euler-Maclaurin Summation can be used to prove (see [10]) that, if we set $C_\alpha(a) = \gamma_\alpha(a) - \frac{\log^\alpha(a)}{a}$ and let $f_\alpha(x) = \frac{\log^\alpha(x+a)}{x+a}$, then we have:

$$\begin{aligned} C_\alpha(a) = &\sum_{r=1}^{m} \frac{\log^\alpha(r+a)}{r+a} - \frac{\log^{\alpha+1}(m+a)}{\alpha+1} - \frac{\log^\alpha(m+a)}{2(m+a)} \\ &- \sum_{j=1}^{\lfloor v/2 \rfloor} \frac{B_{2j}}{(2j)!} f_\alpha^{(2j-1)}(m) + (-1)^{v-1} \int_m^\infty P_v(x) f_\alpha^{(v)}(x)\,dx, \end{aligned}$$

where the B_j denote the Bernoulli numbers (introduced by Bernoulli in [2] of 1713), and P_v is the v-th periodic Bernoulli function (see [13]). This expression has many useful applications; in our recent work, we have used it to find zero-free regions for the fractional derivatives of the Riemann zeta function. There one of the key estimates (Lemma 4.1, [9]) was the bound, for $0 < \alpha \leq 1$,

$$\left| \int_1^\infty P_3(x) f_\alpha'''(x)\,dx \right| < 0.013.$$

Here, with different goals in mind, we will consider another special case of the above Euler-Maclaurin summation formula. We set $m = 1$ and $v = 2$ and analyze the expression

$$C_\alpha(a) = \frac{\log^\alpha(1+a)}{1+a} - \frac{\log^{\alpha+1}(1+a)}{\alpha+1} - \frac{\log^\alpha(1+a)}{2(1+a)} - \frac{1}{12} f_\alpha'(1) - \int_1^\infty P_2(x) f_\alpha''(x)\,dx. \tag{1}$$

2010 *Mathematics Subject Classification.* 11M35.

Now, bounding the generalized fractional Stieltjes constants $\gamma_\alpha(a)$ (or the functions $C_\alpha(a)$), means finding (this time for $1 \le \alpha \in \mathbb{R}$) effective bounds for:

$$\left| \int_1^\infty P_2(x) f_\alpha''(x)\, dx \right|. \tag{2}$$

Since the Bernoulli periodic function $P_2(x)$ involved in the integrals (1) and (2) has a simple Fourier series expansion, established by Hurwitz in 1890 (see [15]) we have

$$P_2(x) = \frac{-n!}{(2\pi i)^n} \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{e^{2\pi i k x}}{k^2} \Bigg|_{n=2} = \frac{2}{(2\pi)^2} \sum_{k=1}^{\infty} \frac{e^{2\pi i k x} + e^{-2\pi i k x}}{k^2} = \frac{4}{(2\pi)^2} \sum_{k=1}^{\infty} \frac{\Re(e^{2\pi i k x})}{k^2}.$$

Because this series is absolutely convergent and $f_\alpha''(x)$ is bounded (see (4)), if we set $S_k := \int_1^\infty e^{2\pi i k x} f_\alpha''(x)\, dx$ and $S^* := \sup_{k\in\mathbb{N}} |S_k|$, then we obtain

$$\left| \int_1^\infty P_2(x) f_\alpha''(x)\, dx \right| = \left| \frac{1}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{k^2} S_k \right| \le \frac{1}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{k^2} |S_k| \le \frac{1}{6} S^*. \tag{3}$$

We obtain effective bounds on $|S_k|$ and S^* by choosing a suitable integration path as done in the method of steepest decent for approximating integrals, see [22, Section II 4] for example. This path originates at a point b on the real axis and then joins a level curve with constant imaginary part that crosses a saddle point and on the right half plane has the asymptote $\frac{\pi}{2} i$.

We find the location of this important saddle point using the Lambert W function (Lambert [14] of 1758), which is the solution of the special case $x = W(x) e^{W(x)}$ of the so-called Lambert transcendental equation (also investigated by Euler in 1783 [7]). Some of these properties will be discussed in the next section.

2. Utility of the Lambert W function

We first rewrite S_k such that it becomes easier to find the saddle point mentioned above. Recall that $f_\alpha(x)$ was defined as $f_\alpha(x) = \frac{\log^\alpha(x+a)}{x+a}$, which means that for its first two derivatives we have $f_\alpha'(x) = \frac{\log^{\alpha-1}(x+a)}{(x+a)^2}(\alpha - \log(x+a))$ and

$$f_\alpha''(x) = \frac{\log^{\alpha-2}(x+a)}{(x+a)^3} \left(\alpha(\alpha-1) - 3\alpha \log(a+x) + 2\log^2(a+x) \right). \tag{4}$$

Now, since $S_k := \int_1^\infty e^{2\pi i k x} f_\alpha''(x)\, dx$, with the change of variables $y = \log(x+a)$ and $b = \log(1+a)$ we can write

$$\begin{aligned} S_k &= \int_1^\infty e^{2\pi i k x} \frac{\log^{\alpha-2}(x+a)}{(x+a)^3} \left(\alpha(\alpha-1) - 3\alpha \log(a+x) + 2\log^2(a+x) \right) dx \\ &= \int_b^\infty e^{2\pi i k (e^y - a)} \frac{y^{\alpha-2}}{e^{3y}} \left(\alpha(\alpha-1) - 3\alpha \cdot y + 2 \cdot y^2 \right) e^y\, dy \\ &= \int_b^\infty e^{2\pi i k (e^y - a) + \alpha \log y} e^{-2y} \left(\frac{\alpha(\alpha-1) - 3\alpha \cdot y + 2\cdot y^2}{y^2} \right) dy. \end{aligned}$$

Let $h_k(y) = 2\pi i k (e^y - a) + \alpha \log y$ and $q(y) = \frac{\alpha(\alpha-1) - 3\alpha \cdot y + 2 \cdot y^2}{y^2}$. Then

$$S_k = \int_b^\infty e^{h_k(y)} e^{-2y} q(y)\, dy. \tag{5}$$

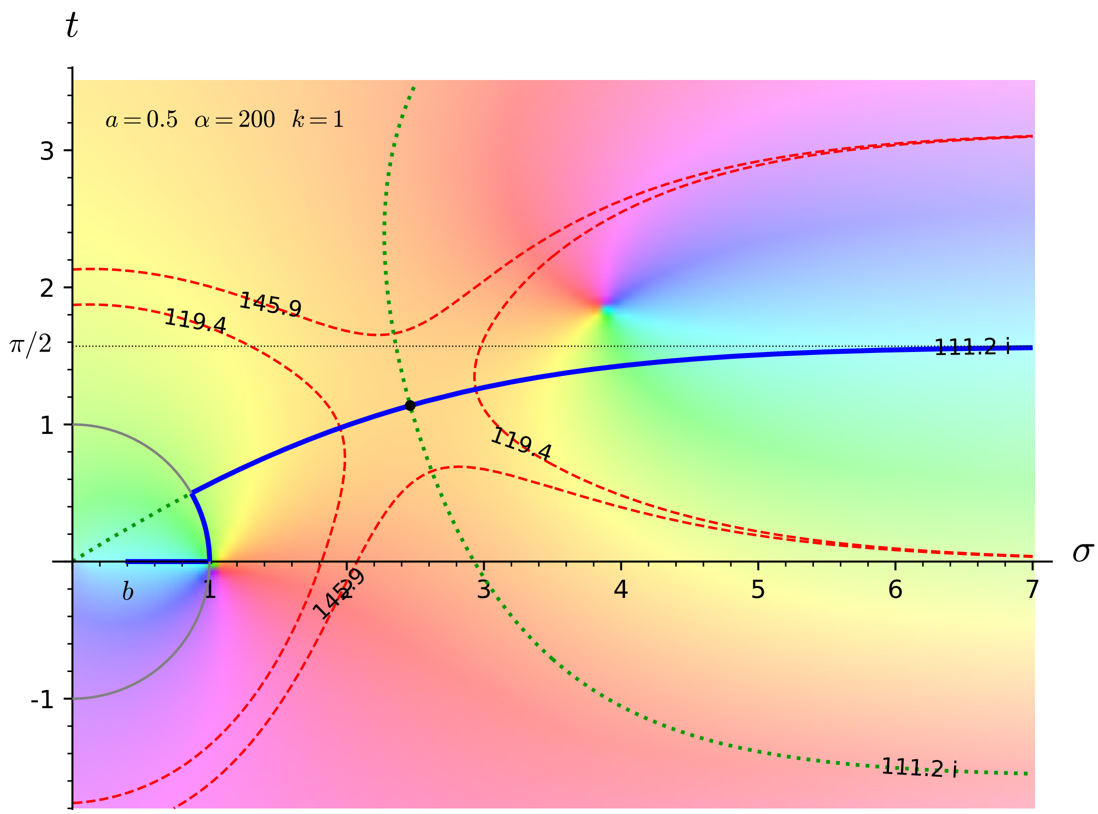

FIGURE 1. The function h_1 for $\alpha = 200$ with the saddle point $\bullet$ near $2.46 + 1.14i$. The dashed red lines are level lines of $\Re(h_1(\sigma + it))$ and the dotted green lines are the level lines where $\Im(h_1(\sigma+it)) = \Im(h_1(w_1(200)))$. The solid blue line is our path of integration.

The function $h_k(y)$ defined above has a saddle point where $h_k'(y_1) = 2\pi ike^{y_1} + \alpha/y_1 = 0$ and the Lambert W function tells us that this happens at $y_1 = W\left(\frac{\alpha i}{2\pi k}\right)$. We use the principal branch W_0 of the Lambert W function and set

$$w_k(\alpha) = W_0\left(\frac{\alpha i}{2\pi k}\right).$$

We make a couple of observations concerning $W_0(it)$ that will be useful later.

Lemma 1. *Let W_0 be the principal branch of the Lambert W function. For $t \in (0, \infty)$, the inverse of $I(t) := \Im(W_0(it))$ is the function $T(y)$, where for $y \in (0, \pi/2)$:*

$$T(y) = \frac{y}{\cos y} \cdot e^{y \cdot \tan(y)}. \tag{6}$$

Proof. Considering the real part of $W_0(it) \cdot e^{W_0(it)} = it$ we get $\Re(W_0(it)) = I(t) \cdot \tan(I(t))$, so that $W_0(it) = I(t)(\tan(I(t)) + i)$. Using this in $W_0(it) \cdot e^{W_0(it)} = it$, and considering only the imaginary parts, we obtain $I(t) \cdot e^{I(t)\tan(I(t))}\frac{1}{\cos(I(t))} = t$. This shows that, for $y \in [0, \pi/2)$, if we set $T(y) = \frac{y}{\cos y} \cdot e^{y \cdot \tan(y)}$, then T is the inverse of I. □

It follows immediately from Lemma 1 that $T(0) = 0$ and $\lim_{y\to\pi/2} T(y) = \infty$ and that

$$T'(y) = \left(((\tan^2(y) + 1) \cdot y + \tan(y)) \cdot y + y \tan y + 1\right) \frac{e^{y \tan(y)}}{\cos(y)}.$$

Hence $T'(y) > 0$ for $y \in [0, \pi/2)$, and therefore $I(t) > 0$, for $t \in (0, \infty]$. Thus for $t > 0$ we have

$$0 < I(t) < \frac{\pi}{2} \tag{7}$$

and $\lim_{t\to\infty} I(t) = \pi/2$. This implies

$$\Re(W_0(it)) = I(t) \cdot \tan(I(t)) > 0. \tag{8}$$

Taking the logarithm of $W_0(it) \cdot e^{W_0(it)} = it$ and only considering real parts we obtain $\Re(W_0(it)) = \log(t) - \Re(\log W_0(it))$. Thus, for $t > 1.97$ where $|W_0(it)| > 1$,

$$\Re(W_0(it)) < \log(t). \tag{9}$$

We will employ the following two lemmas to show that we can set $S^* = |S_1|$. (The second of these results also shows why, everywhere below, we will assume $\alpha \geq 2\pi$.)

Lemma 2. *For $t > 0$ we have $\frac{d}{dt}\left(\Re\left(\log(W_0(it)) - \frac{1}{W_0(it)}\right)\right) > 0$.*

Proof. With $W_0'(x) = \frac{W_0(x)}{x \cdot (1+W_0(x))}$ we get $\frac{d}{dt}\Re\left(\log(W_0(it) - \frac{1}{W_0(it)}\right) = \frac{1}{t}\frac{\Re W_0(it)}{|W_0(it)|} > 0$, as desired. ☐

Lemma 3. *For $k \in \mathbb{N}$ and $\alpha \in [2\pi, \infty)$ we have $k \cdot \sin(\Im(w_k(\alpha))) \geq 1/2$.*

Proof. Representing the cosine and tangent functions in (6) by the sine function, and using that $\sin^{-1} x \leq \frac{\pi}{3}x$, for $x \in \left[0, \frac{1}{2}\right]$, we deduce:

$$\begin{aligned}
T\left(\sin^{-1}\frac{1}{2k}\right) &= \frac{\sin^{-1}\frac{1}{2k}}{\cos\left(\sin^{-1}\frac{1}{2k}\right)} \cdot e^{\left(\sin^{-1}\frac{1}{2k}\right)\cdot\tan\left(\sin^{-1}\frac{1}{2k}\right)} \\
&= \frac{\sin^{-1}\frac{1}{2k}}{\sqrt{1-\sin^2\left(\sin^{-1}\frac{1}{2k}\right)}} \cdot e^{\left(\sin^{-1}\frac{1}{2k}\right)\cdot\frac{\sin\left(\sin^{-1}\frac{1}{2k}\right)}{\sqrt{1-\sin^2\left(\sin^{-1}\frac{1}{2k}\right)}}} \\
&\leq \frac{\frac{\pi}{6k}}{\sqrt{1-\frac{1}{4k^2}}} \cdot e^{\frac{\pi}{6k}\cdot\frac{\frac{1}{2k}}{\sqrt{1-\frac{1}{4k^2}}}} \leq \frac{\pi}{3\sqrt{4k^2-1}} \cdot e^{\frac{\pi}{6k\sqrt{4k^2-1}}} \\
&< \frac{\pi}{3\sqrt{3}k} \cdot e^{\frac{\pi}{6\sqrt{3}k^2}} < 0.61 \cdot \frac{1}{k} \cdot 1.4 < \frac{1}{k}.
\end{aligned}$$

Applying the functions I and sine to both sides of this inequality yields

$$\frac{1}{2k} \leq \sin\left(I\left(\frac{1}{k}\right)\right).$$

Because of the monotonicity of I for $\alpha \geq 2\pi$, we get $\frac{1}{2} \leq k \sin\left(I\left(\frac{\alpha}{2\pi k}\right)\right) = k \cdot \sin\left(\Im(w_k(\alpha))\right)$. ☐

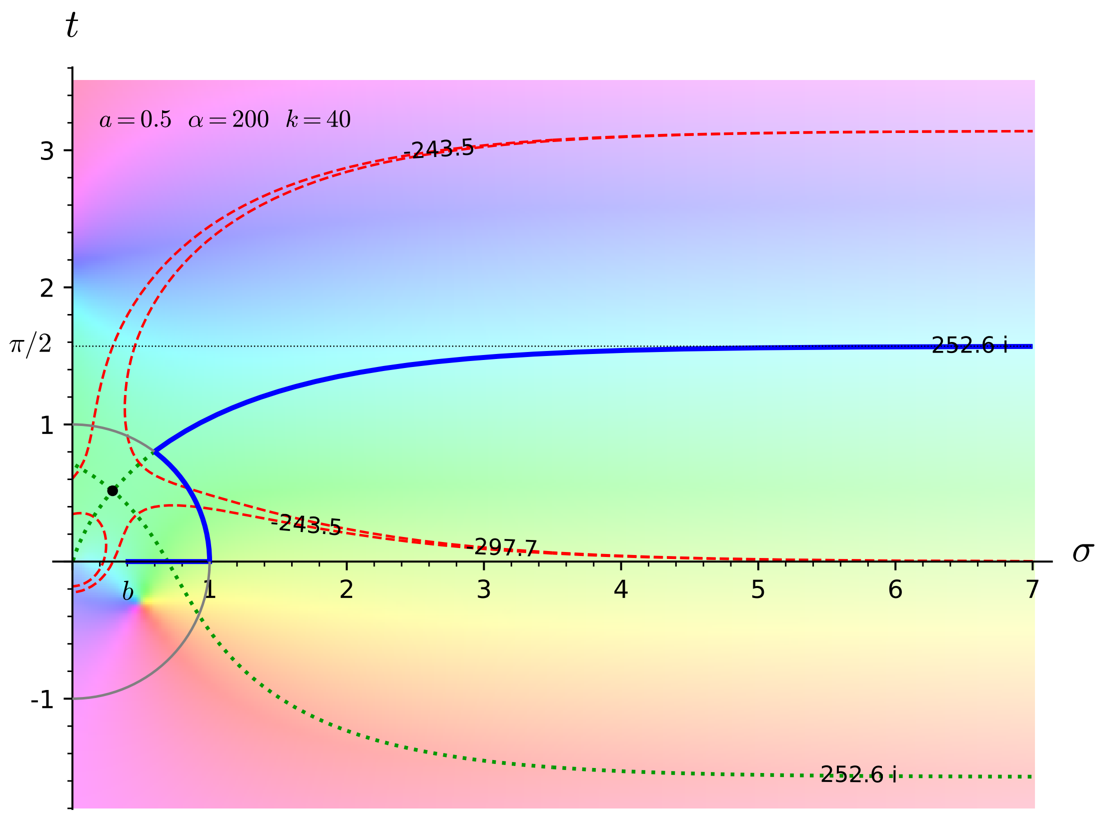

FIGURE 2. The function h_{40} for $\alpha = 200$ with the saddle point $\bullet$ near $0.29 + 0.52i$. The dashed red lines are level lines of $\Re(h_{40}(\sigma + it))$ and the dotted green lines are the level lines where $\Im(h_{40}(\sigma + it)) = \Im(h_{40}(w_{40}(200)))$. The solid blue line is our path of integration.

3. Bounding the Integrals

First, recall the definitions of the quantities $|S_k|$ we are interested in:

$$|S_k| := \left| \int_b^\infty e^{h_k(y)} e^{-2y} q(y)\, dy \right|,$$

where (as in Lemma 1) $h_k(y) = 2\pi i k(e^y - a) + \alpha \log y$, $q(y) = \frac{\alpha(\alpha-1) - 3\alpha \cdot y + 2 \cdot y^2}{y^2}$ with (as in Lemma 3) $\alpha \geq 2\pi$, and $b = \log(1 + a)$. We evaluate S_k by integrating along the contour made up of 4 components c_1, c_2, c_3, and c_4 where

- c_1 starts at $b \in \mathbb{R}^+$ and goes along the positive real axis to 1,
- c_2 starts at 1 and follows the unit circle until the point u where $\Im(h_k(u)) = \Im(h_k(w_k(\alpha))$,
- c_3 starts at the end point u of c_2 and follows the level line $\Im(h_k(y)) = \Im(h_k(w_k(\alpha))$ until the point v with $\Re(v) = 2 \log \alpha$ is reached,
- c_4 starts at the end point v of c_3 and approaches infinity on the level line $\Im(h_k(y)) = \Im(h_k(w_k(\alpha))$.

From our observations (7) and (8) we know that $0 < \Im(w_k(\alpha)) < \pi/2$ and $\Re(w_k(\alpha)) > 0$. Setting $\Im(h_k(y)) = \Im(h_k(w_k(\alpha)))$ we see that as $\Re(y) \to \infty$ we have $\Im(y) \to \frac{\pi}{2}$. On this level line, from the origin to the saddle at $w_k(\alpha)$, the value of $\Re(h_k(y))$ strictly increases, and after crossing it the value of $\Re(h_k(y))$ strictly decreases. Figures 1 and 2 provide illustrative examples of this.

In the next four lemmas we will estimate the integral for each of these four components separately.

Lemma 4. *Let* $L_1 := \int_{c_1} e^{h_k(y)} e^{-2y} q(y)\, dy$*, then* $|L_1| < \alpha + 3 + \frac{1}{\pi}$.

Proof. Let $y \in \mathbb{R}^+$. First, let us note that

$$\Re(h_k(y)) = \Re(2\pi i k(e^y - a) + \alpha \log(y)) = -2\pi k e^{\Re(y)} \sin \Im(y) + \alpha \log |y| \tag{10}$$

and because $|e^{-2y}| \leq 1$, we can write:

$$\begin{aligned}
|L_1| &= \left| \int_b^1 e^{h_k(y)} e^{-2y} q(y)\, dy \right| \leq \int_b^1 |e^{h_k(y)}| \cdot |q(y)|\, dy \\
&\leq \int_b^1 e^{-2\pi k e^{\Re(y)} \sin \Im(y) + \alpha \log |y|} \cdot |q(y)|\, dy \leq \int_b^1 e^{\alpha \log y} \cdot |q(y)|\, dy \\
&\leq \int_b^1 y^\alpha \left(\frac{\alpha(\alpha-1)}{y^2} + \frac{3\alpha}{y} + 2 \right) dy = \left[\alpha y^{\alpha-1} + 3y^\alpha + \frac{2y^{\alpha+1}}{\alpha+1} \right]_b^1 \\
&= \left(\alpha + 3 + \frac{2}{\alpha+1} \right) - \left(\log(a+1)^{\alpha-1} + 3\log(a+1)^\alpha + \frac{2\log(a+1)^{\alpha+1}}{\alpha+1} \right) < \alpha + 3 + \frac{1}{\pi}. \quad \square
\end{aligned}$$

Lemma 5. *Let* $L_2 := \int_{c_2} e^{h_k(y)} e^{-2y} q(y)\, dy$*, then* $|L_2| < (\alpha^2 + 2\alpha + 2)\frac{\pi}{2}$.

Proof. Recall that c_2 starting at 1 follows the unit circle until the point u with $\Im(h_k(u)) = \Im(h_k(w_k(\alpha))$ is reached. First observe that on the unit circle we have $|y| = 1$ and $0 \leq \Im(y) \leq 1$, and thus, by (10),

$$\Re(h_k(y)) = -2\pi k e^{\Re(y)} \sin \Im(y) + \alpha \log |y| = -2\pi k e^{\Re(y)} \sin \Im(y) \leq 0. \tag{11}$$

Moreover, we can write

$$|q(y)| \leq \left| \frac{\alpha(\alpha-1)}{y^2} \right| + \left| \frac{3\alpha}{y} \right| + 2 = \frac{\alpha(\alpha-1)}{|y^2|} + \frac{3\alpha}{|y|} + 2 = \alpha^2 + 2\alpha + 2. \tag{12}$$

Since replacing u by i as the endpoint of the contour c_2 can only extend the path of integration, the bounds (11) and (12) give:

$$|L_2| \leq \int_{c_2} |e^{h_k(y)}| \cdot |e^{-2y}| \cdot |q(y)|\, dy \leq \int_{c_2} |q(y)|\, dy \leq (\alpha^2 + 2\alpha + 2)\frac{\pi}{2}. \quad \square$$

Lemma 6. *Let* $L_4 := \int_{c_4} e^{h_k(y)} e^{-2y} q(y)\, dy$*, then* $|L_4| < \alpha^2 + 2\alpha + 2$.

Proof. Our integration path c_4 follows the level line with $\Im(y) = \Im(w_k(\alpha))$ starting at the point v with $\Re(v) = 2\log\alpha$. Here we have $|y| \geq 1$ and thus, as in (12) we have $|q(y)| \leq \alpha^2 + 2\alpha + 2$. We get

$$|L_4| \leq \int_{c_4} |e^{h_k(y)-2y}|(\alpha^2 + 2\alpha + 2)\, dy = (\alpha^2 + 2\alpha + 2) \int_{c_4} e^{\Re(h_k(y)-2y)}\, dy. \tag{13}$$

By (9) we have $\Re(w_k(\alpha)) < \log(\alpha)$. So v lies to the right of the saddle. For $\Re(y) \geq 2\log\alpha$ we have

$$\Re(h_k(y)) = -2\pi k e^{\Re(y)} \sin \Im(y) + \alpha \log |y| \leq -\pi e^{\Re(y)} + \alpha \log |y| < 0. \tag{14}$$

To see this, just note that in our region $\Re(y) > 2\log\alpha$ we have (i) $e^{\Re(y)/2} > \alpha$, while at the same time, becuase of the concavity of both the logarithmic and the Lambert W functions, we can write $\log\log|y| = \log\frac{1}{2} + \log\log|y|^2 = \log\frac{1}{2} + \log\log(\Re(y)^2 + \Im(y)^2) < \log\frac{1}{2} + \log\log(2\Re(y)^2) < \log\frac{1}{2} + \log\log 2 + \log 2 + \log\log\Re(y) < \Re(y)/2$, which implies (ii) $e^{\Re(y)/2} > \log|y|$. Multiplying (i) and (ii) together yields $e^{\Re(y)} > \alpha\log|y|$, which implies (14). Finally, plugging the bound (14) into the integral in (13) and $\Re(y) \geq 2\log\alpha$ give:

$$|L_4| \leq (\alpha^2 + 2\alpha + 2)\int_{c_4}\left|e^{-\pi e^{\Re(y)} + \alpha\log|y|}\right| e^{\Re(-2y)}\,dy \leq (\alpha^2 + \alpha + 2)\int_{c_4} e^{\Re(-2y)}\,dy < \alpha^2 + 2\alpha + 2. \square$$

Lemma 7. *Let* $L_3 := \displaystyle\int_{c_3} e^{h_k(y)}e^{-2y}q(y)\,dy$, *then*

$$|L_3| < e^{\Re\left(-\frac{\alpha}{w_k(\alpha)} + \alpha\log w_k(\alpha)\right)}(\alpha^2 + 2\alpha + 2)(2\log\alpha + \pi/2).$$

Proof. On the level line $\Im(h_k(y)) = \Im(h_k(w_k(\alpha))$ the function $h_k(y)$ has its (real) maximum at the saddle point $w_k(\alpha)$. This allows us to bound the real part of $h_k(y)$ as:

$$\Re(h_k(y)) \leq \Re(h_k(w_k(\alpha)) = \Re\left(2\pi ik(e^{w_(\alpha)} - a) + \alpha\log w_k(\alpha)\right) = \Re\left(-\frac{\alpha}{w_k(\alpha)} + \alpha\log w_k(\alpha)\right),$$

With the above and (12) we get

$$\begin{aligned}
|L_3| &\leq \int_{c_3}\left|e^{h_k(y)}\right|\cdot\left|e^{-2y}\right|\cdot|q(y)|\,dy \\
&\leq e^{\Re\left(-\frac{\alpha}{w_k(\alpha)} + \alpha\log w_k(\alpha)\right)}(\alpha^2 + 2\alpha + 2)\int_{c_3}\left|e^{-2y}\right|\,dy \\
&\leq e^{\Re\left(-\frac{\alpha}{w_k(\alpha)} + \alpha\log w_k(\alpha)\right)}(\alpha^2 + 2\alpha + 2)(2\log\alpha + \pi/2),
\end{aligned}$$

where the last inequlity holds because $\left|e^{-2y}\right| < 1$ on c_3 and because the integral for the arc-length of the curve c_3 can be bounded from above by the sum of lenghts of two adjacent sides of the rectangle that contains it. $\square$

Putting these four Lemmas together, we immediately get the following bound:

$$\begin{aligned}
&|S_k| \leq |L_1| + |L_2| + |L_3| + |L_4| \\
&< \alpha + 3 + \frac{1}{\pi} + (\alpha^2 + 2\alpha + 2)\left(1 + \frac{\pi}{2} + e^{\Re\left(-\frac{\alpha}{w_k(\alpha)} + \alpha\log w_k(\alpha)\right)}(2\log\alpha + \pi/2)\right).
\end{aligned} \tag{15}$$

4. The Final Bound

Now we can prove the following general result:

Theorem 8. *For* $\alpha \geq 2\pi$ *and* $a \in (0, 1]$ *let us denote by* $\gamma_\alpha(a)$ *the fractional Stieltjes constants and write* $C_\alpha(a) = \gamma_\alpha(a) - \frac{\log^\alpha a}{a}$. *If we set* $w(\alpha) := W_0\left(\frac{\alpha i}{2\pi}\right)$, *where* W_0 *is the principal branch of the Lambert* W *function, then*

$$|C_\alpha(a)| < \alpha^2 + \frac{3}{4}\alpha^2\log\alpha\cdot\left|e^{\alpha(\log w(\alpha) - 1/w(\alpha))}\right|.$$

Proof. From Lemmas 2 and 6 it follows that the quantities S_k decrease as k increases. Thus we can set $S^* = |S_1|$ in (3), and with the help of the bound (15) we can rewrite (1) as:

$$\begin{aligned}
|C_\alpha(a)| &< \log^\alpha(1 + a)\left|\frac{1}{2}\frac{1}{1 + a} - \frac{\log(1 + a)}{\alpha + 1} + \frac{1}{12(1 + a)^2}\right| + \left|\frac{1}{12}\frac{\log^{\alpha - 1}(1 + a)}{(1 + a)^2}\alpha\right| + \frac{1}{6}|S_1| \\
&< 1 + \frac{1}{12}\alpha + \frac{1}{6}\left[\alpha + 3.3 + (\alpha^2 + 2\alpha + 2)\left(\left(1 + \frac{\pi}{2}\right) + (2\log\alpha + \pi)\cdot e^{\Re\left(-\frac{\alpha}{w_1(\alpha)} + \alpha\log w_1(\alpha)\right)}\right)\right]
\end{aligned}$$

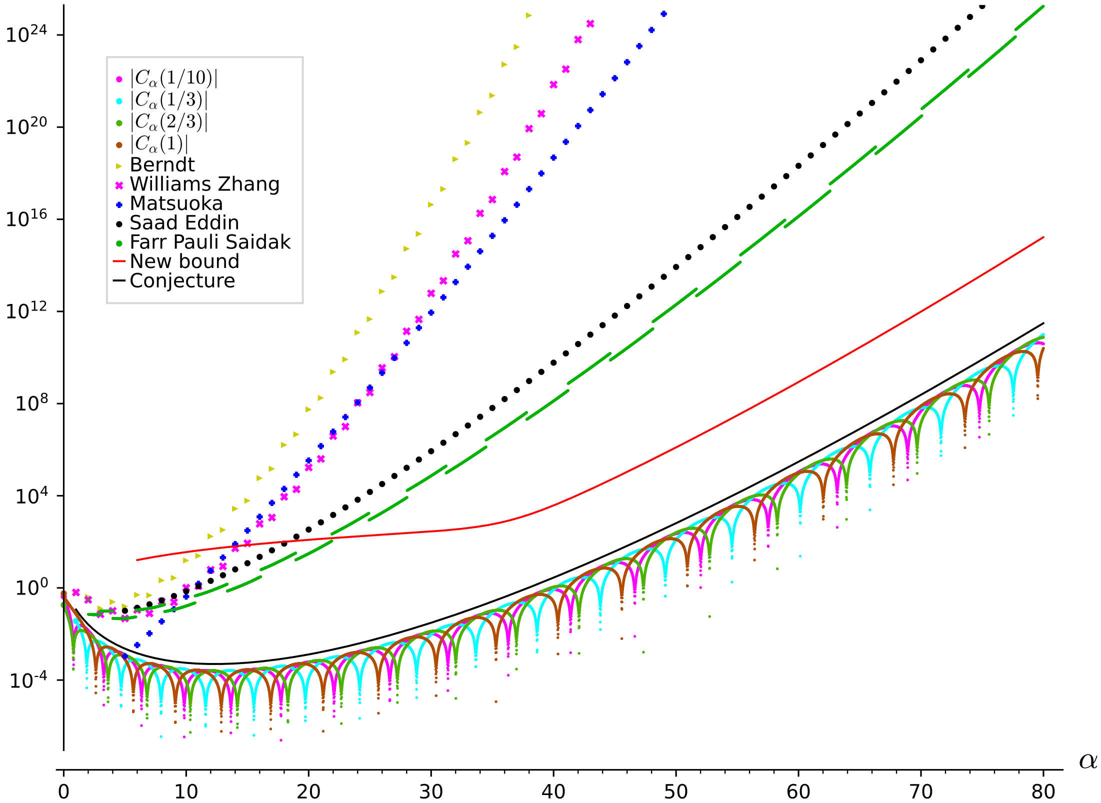

FIGURE 3. On a logarithmic scale we show the absolute values of the Stieltjes constants $\gamma_\alpha = C_\alpha(1)$, $C_\alpha(2/3)$, $C_\alpha(1/3)$ and $C_\alpha(1/10)$ along with the bounds by Berndt [1], Williams and Zhang [21], Matsuoka [16] and Saad Eddin [19], our bound and conjecture from [10] as well as our new bound from Theorem 8.

$$< 1.55 + \frac{\alpha}{4} + \frac{1}{2}\left(\alpha^2 + 2\alpha + 2\right) + \frac{1}{6}\left(\frac{3}{2}\alpha^2\right)(3\log\alpha)\cdot e^{\Re\left(-\frac{\alpha}{w_1(\alpha)} + \alpha\log w_1(\alpha)\right)}$$
$$< \alpha^2 + \frac{3}{2}\alpha^2\log\alpha \cdot \left|e^{\alpha(\log w(\alpha) - 1/w(\alpha))}\right|,$$

since $\pi < 4\log\alpha$ and $\frac{1}{6}(1+\frac{\pi}{2}) < \frac{1}{2}$, and in our range we also have $2.55 + \frac{5}{4}\alpha < \frac{1}{2}\alpha^2$. $\square$

Note that the main term of the bound in Theorem 8 differs only by a factor of $\alpha^2\log\alpha$ from the conjectured bound given in [10]:

$$|C_\alpha(a)| \le 2\left|e^{\alpha(\log w(\alpha) - 1/w(\alpha))}\right|. \tag{16}$$

In Figure 3 we compare Theorem 8 and (16) with previously known bounds for $\gamma_\alpha = C_\alpha(1)$. For $m \in \mathbb{N}$ we have:

(1) the bound by Berndt [1]: $|\gamma_m| \le \frac{(3+(-1)^m)(m-1)!}{\pi^m}$
(2) the bound by Williams and Zhang [21]: $|\gamma_m| \le \frac{(3+(-1)^m)(2m)!}{m^{m+1}(2\pi)^m}$
(3) the bound by Matsuoka [16] which holds for $m > 4$: $|\gamma_m| < 10^{-4}(\log m)^m$

(4) the bound by Saad Eddin [19]: Let $\theta(m) = \frac{m+1}{\log \frac{2(m+1)}{\pi}} - 1$ then

$$|\gamma_m| \leq m! \cdot 2\sqrt{2} e^{-(n+1)\log\theta(m) + \theta(m)\left(\log\theta(m) + \log\frac{2}{\pi e}\right)} \left(1 + 2^{-\theta(m)-1} \frac{\theta(m)+1}{\theta(m)-1}\right).$$

(5) our bound from [10]: For $\alpha \in (0, \infty)$ let $x = \frac{\pi}{2} e^{W_0\left(\frac{2(\alpha+1)}{\pi}\right)}$ then

$$|\gamma_\alpha| \leq \frac{(3 + (-1)^{n+1})\Gamma(\alpha+1)}{(2\pi)^{n+1}(n+1)^{\alpha+1}} \frac{(2(n+1))!}{(n+1)!} \text{ where } n = \begin{cases} \lfloor x \rceil & \text{if } x < \alpha \\ \lceil \alpha - 1 \rceil & \text{otherwise} \end{cases}$$

Remark. With (9) we get $\Re(\log(w(\alpha)) - 1/w(\alpha)) < \Re(\log w(\alpha)) < \log\log(\alpha)$. Hence

$$\left| e^{\alpha(\log w(\alpha) - 1/w(\alpha))} \right| = e^{\alpha \Re(\log w(\alpha) - 1/w(\alpha))} < (\log \alpha)^\alpha$$

Thus the main term of Matsuoka's bound follows from (16).

5. Acknowledgments

We'd like to thank the anonymous referee for several useful comments and for pointing out a couple of miscalculations. All plots were created with the computer algebra system SageMath [18].

References

[1] Berndt, C.: *On the Hurwitz zeta-function*, Rocky Mountain J. Math., Vol. 2, No. 1, 151–157, 1972.

[2] Bernoulli, J: *Ars Conjectandi*, Part II, Chap. III, Thurneysen Bros., Basel, 1713.

[3] Coffey, M. W.: *Functional equations for the Stieltjes constants*, Ramanujan J., 39 (2016), no. 3, 577–601.

[4] Coffey, M. W., Knessl, C.: *An effective asymptotic formula for the Stieltjes constants*, Math. Comp., 80 (2011), no. 273, 379–386.

[5] Euler, L.: *Variae observationes circa series infinitas*, Comment. Acad. Scient. Petropol., Vol. 9, 160–188, 1744.

[6] Euler, L.: *Remarques sur un beau rapport entre les séries des puissances tant directes que réciproques*, Mémoires de l'académie des sciences de Berlin, Vol. 17, 83–106, 1768.

[7] Euler, L.: *De serie Lambertina Plurimisque eius insignibus proprietatibus*, Acta Acad. Scient. Petropol., 2, 29–51, 1783.

[8] Farr, R., Pauli, S., Saidak, F.: *On Fractional Stieltjes Constants*, Indag. Math., Vol. 29, No. 5, 1425–1431, 2018.

[9] Farr, R., Pauli, S., Saidak, F.: *A Zero free region for the fractional derivatives of the Riemann zeta function*, New Zealand J. Math., Vol. 50, 1–9, 2020

[10] Farr, R., Pauli, S., Saidak, F.: *Approximating and Bounding fractional Stieltjes constants*, Funct. Approx. Comment. Math, Vol. 64, No. 1, 7–22, 2021.

[11] Hurwitz, A.: *Einige Eigenschaften der Dirichletschen Funktionen $F(s) = \sum(\frac{D}{n}) \cdot \frac{1}{n^s}$, die bei der Bestimmung der Klassenanzahlen Binärer quadratischer Formen auftreten*, Z. für Math. und Physik, 27, 86–101, 1882.

[12] Kreminski, R.: *Newton-Cotes integration for approximating Stieltjes (generalized Euler) constants*, Math. Comp., Vol. 72, 1379: 1397. 2003.

[13] Kimball, B.: *A generalization of the Bernoulli polynomial of order one*, Bull. Amer. Math. Soc. Vol. 41, No. 12, 894–900, 1935.

[14] Lambert, J. H. *Observations variae in Mathesin Puram,* Acta Helvitica, Phys-Math., 3, 128–168, 1758.

[15] Lehmer, D. H.: *A New Approach to Bernoulli Polynomials*, American Math. Monthly, Vol. 95, No. 10, 905–911, 1988.

[16] Matsuoka, Y. : *Generalized Euler constants associated with the Riemann zeta function*, Number Theory and Combinatorics: Japan 1984, World Scientific, Singapore, pp. 279–295, 1985.

[17] Riemann B.: *Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse*, Monatsberichte der Berliner Akademie, 6 pp., November 1859.

[18] The Sage Developers: *SageMath, the Sage Mathematics Software System (Version 9.2)*, `http://www.sagemath.org`, 2020.

[19] Saad Eddin, S.:, Explicit upper bounds for the Stieltjes constants, J. Number Theory 133 (2013), no. 3, 1027-1044.

[20] Stieltjes, T. J.: *Correspondence d'Hermite et de Stieltjes*, Tomes I, II, Gauthier-Villars, Paris, 1905.

[21] Williams, K. S., Zhang, N. Y.: *Some results on the generalized Stieltjes constants.* Analysis 14 (1994), no. 2: 3, 147–162.

[22] Wong, K.: *Asymptotic Approximations of Integrals*, SIAM, 2001.

Department of Mathematics and Statistics, UNC Greensboro, Greensboro, NC 27402, USA
Email address: s_pauli@uncg.edu, f_saidak@uncg.edu